

Simple laws of urban growth

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By analysing the evolution of the street network of Greater London from the late 1700s to the present, we are able to shed light on the inner mechanisms that lie behind the growth of a city. First we define an object called a *city* as a spatial discontinuous phenomena, from clustering the density of street intersections. Second, we find that the city growth mechanisms can be described by two logistic laws, hence can be determined by a simple model of urban network growth in the presence of competition for limited space.

The study of the city in classical physical and geometric terms has preoccupied urban researchers since the time of Newton if not before [1]. Advances in digital technologies that enable us to capture and represent data and explore simulations of urban structure and growth have advanced rapidly in the last two decades and GIS (geographic information system) technologies, in particular, have given us the chance to analyse big datasets such as street networks. Already important results have been produced with regard to the physical properties of the city [2]. Still many problems remain open, especially regarding the pattern formation of city infrastructures and although many global statistical laws involving the scaling of city size and population densities have been known for a long time, there is still little knowledge about the microscopic mechanisms that generate such complexity.

Here we pursue these ideas through the analysis of a unique dataset that is able to shed light on many unanswered questions. This dataset is based on mapping the street patterns of London defined for the Greater London Authority area (or GLA hereafter) from 1786 to 2010 (see Fig.1). Each street or road segment is classified according to a four level hierarchy based on motorways, class A, class B, and minor roads. Documenting the evolution of the road network in 9 time-series enables us to isolate the core of the city which we call *London*, for each of the time slices, from examination of the clustering of streets that determine its physical extent.

A city of course is composed by many layers of infrastructure which underpin its social and economic functioning. These are interconnected and coevolve, and lead to many different definitions of its physical extent. The definition of a *city* can thus be quite blurred with respect to these layers. Nevertheless a precise definition is crucial to any statistical analysis [3, 4]. Cities can be analysed within their administrative boundaries, or in terms of their population densities [5], but here we need to define an object that we call a *city* in such a way that there are some measurable properties that clearly define its presence. Thus we should expect some transition of these

measurable quantities defined for the area where the *city* is present, to the larger area where the *city* is not present. Otherwise if the city were some continuous phenomena, its boundaries, as its definition, would be arbitrarily defined by a given threshold.

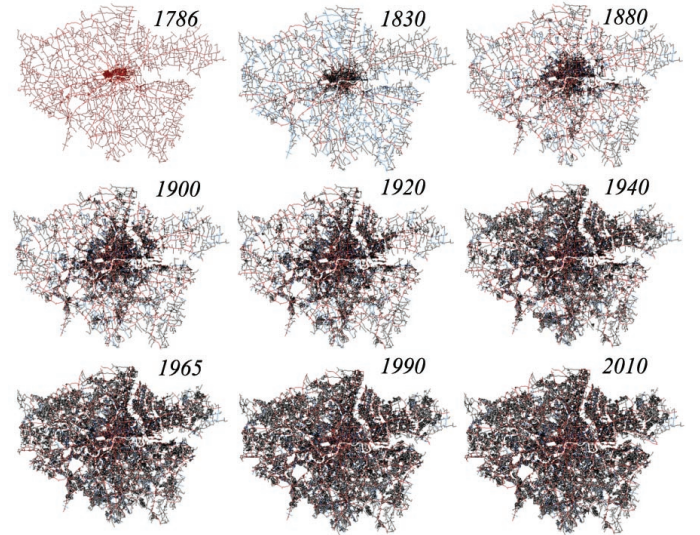


FIG. 1. The street network in the GLA (Greater London Area) from 1786 to 2010. Different colors of the roads correspond to different road classifications (red A roads and motorways, blue B roads, gray minor roads).

In order to analyse the dataset shown in Fig.1 we need first to define a city at the level of its basic infrastructures, i.e. in terms of its street network. City growth as a street network can be understood as the coevolution of two distinct phenomena, based on the hierarchy of its roads. On the one hand, we have the growth of major roads (including motorways, class A and class B roads) and, on the other, the growth of minor roads. A and B roads represent the backbone of the city where the main flows of people and materials sustaining the city take place. From our analysis, it is clear that the main structure of such a backbone pre-existed the city in

1786, although additions to this backbone have increased by a factor of 4.7 during the subsequent 224 years. Minor roads divide the blocks created by the A and B roads into smaller areas, and these are mainly devoted to local residential and business use. However these minor roads comprise most of the urbanisation that has taken place during the last 224 years, increasing by a factor of 13.3 during this time. During this period of city growth, this mixture of growth of major and minor roads leads to a considerable fragmentation of the major roads, thus to an increasing number of intersections, and finally to the generation of the complex street network that we experience in large cities [6, 7].

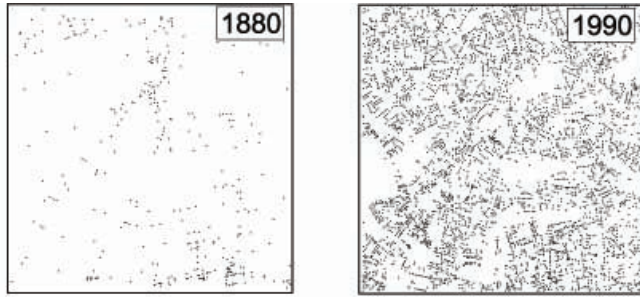


FIG. 2. Left panel: Street intersections inside a 10 km side square in the non-urbanised area of the GLA in 1880. Right panel: Street intersections inside of the same square in 1990.

The fact that the initial development of the A and B roads mostly precedes the development of minor roads is clear from Fig.1 where a dense net of A and B roads is already present in 1786. The city then develops by filling the land parcels generated by A and B roads with minor roads, for most development tends to be on a smaller scale than the street segments themselves. These two different phenomena are reflected in different street intersection patterns depicted in Fig.2, which shows the same 10km x 10km grid square at a interval of around one century. In the left panel we show the intersection pattern of the non-urbanised area of the 1880 GLA map where there are $N = 355$ points (intersections) with an average density $\rho = 3.55 \cdot 10^{-6} m^{-2}$. If the system were homogeneous, this density would suggest a length-scale of the order $\lambda \sim 1/\sqrt{\rho} \approx 530m$, one that we generally experience when we are outside the city. In the right panel, we show the intersections for the same grid square in 1990 where there are $N = 4704$ intersections with an average density $\rho = 4.7 \cdot 10^{-5} m^{-2}$. This gives a scale to the system of the order of $\lambda \approx 146m$, that is a spatial scale more like that which we experience in our daily urban life.

These simple observations relate the identification of an urban area to the street intersection density in that area. To test this idea, we show a qualitative analysis

in Fig.3 of the intersection density in the GLA area. A first glance at this figure suggests the spreading of a wild fire, or more generally of a percolation phenomena [8]. If we then use box-counting to calculate the number of boxes $N^B(R)$ at scale R that are occupied by intersections, we find that this quantity scales as a power law $N^B(R) \propto R^{-D_F}$ for fractal objects, where D_F is the fractal dimension [9].

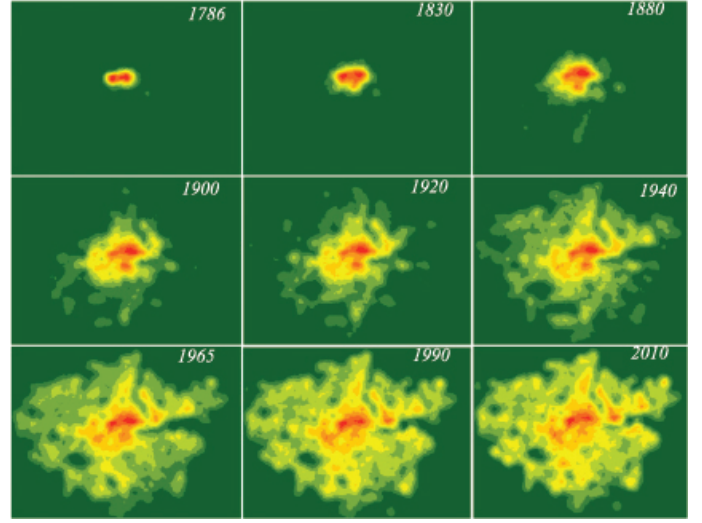


FIG. 3. Street intersection density surfaces in the GLA area from 1786 to 2010.

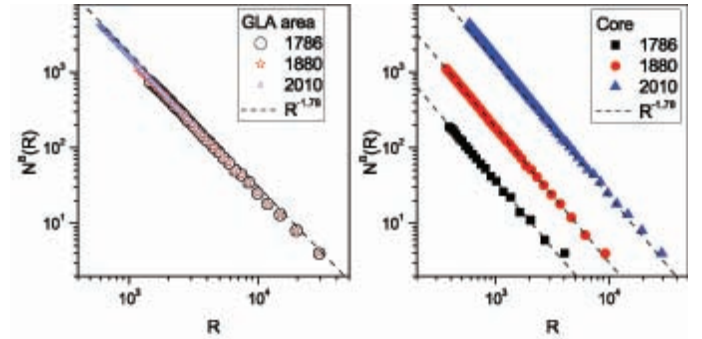


FIG. 4. Number of occupied squares $N^B(R)$ at scale R for the intersection density maps and measurement of the fractal exponent D_F . Left panel: For the GLA. Right panel: For the density core we define as *London*.

In the left panel of Fig.4, we show $N^B(R)$ for three street patterns at time slices, each separated by about 1 century for the GLA area. The relationships overlap, closely fitting a power law with a fractal exponent $D_F \approx 1.78$. In the right panel of the same figure, we carry out an identical analysis but for the *city* area of London (which we define below as the *core*). Interestingly the behaviour does not change at these different scales which is a property of fractal objects, but the growth pattern is clearly different in the core from the GLA area. This

analysis shows that the intersection density is a robust property of the infrastructural system of the GLA and gives us confidence in testing so that we can define the boundary of the *city* or *core*.

In order to split the core from the wider area, that is the area of high intersection density patterns from the parts of the maps where the intersection density is low, we apply the *Jenks natural breaks algorithm* to the GLA density maps [10]. This methodology begins with a certain number of classes, then generates regions minimising the within-class variance, while maximising the inter-class variance between different regions. Setting the number of classes for the intersection density equal to two, we find that the algorithm clearly identifies urbanised areas from areas that are not urbanised. We then use the boundaries of these regions as those of the city. In this way, the spread of urbanization in the GLA area is well-defined and the city boundary for London is clearly demarcated through the years as we show in Fig. 5.

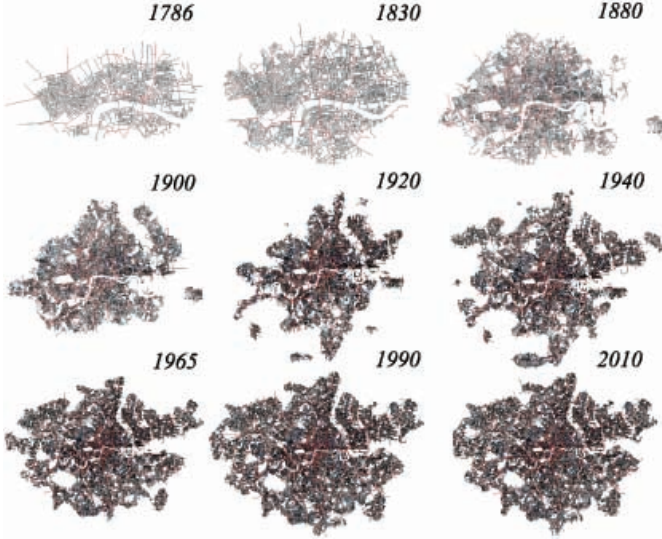


FIG. 5. The street network for each time period derived from the GLA area using the Jenks clustering algorithm.

We refer to the urbanised regions of Fig.5 as “*London*”, in order to distinguish them from the GLA area in Fig.1. Such a choice enables us to identify the emergence of some robust properties that identify the city as a well-defined physical object. These properties are shown in Fig.6. In the top left panel, we show the street length distribution $P(l)$ for the street segments defined by every two consecutive intersections. In the left panel, the measure is calculated for the GLA area in 1786, when it is mostly not yet urbanised. The distribution displays a clear exponential tail. In the right panel, we show the same measure computed for three time slices which differ by about one century for London. In this case, the distribution is robustly log-normal throughout the 224 years.

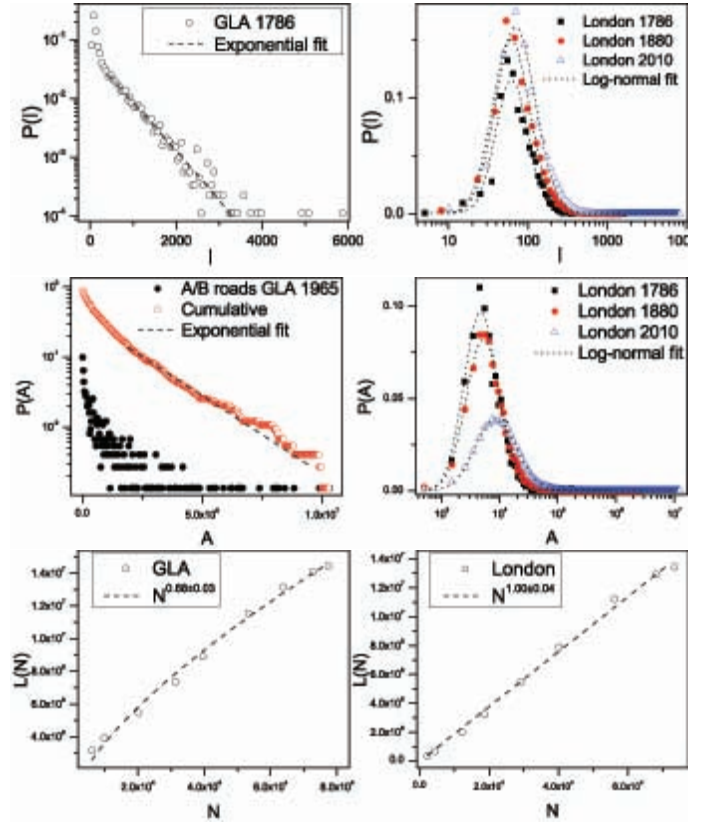


FIG. 6. Top-left panel: Street length distribution as measured in the GLA area in 1786. Top-right panel: Street length distribution for London as defined by the Jenks’ algorithm about one century apart. Middle-left panel: Parcel area distribution for the network generated by A and B roads in the 1965 GLA area. Middle-right panel: Parcel area distribution for London. Bottom-left panel: Total street length as a function of the number of intersections for the GLA area. Bottom-right panel: Total street length as a function of the number of intersections for London.

In the middle right panel of Fig.6, we show the face area distribution $P(A)$ for the 1786, 1880 and 1900 London street intersection maps. The plots are also well fitted by log-normal distributions. This is a robust property of the system as defined by the minor roads. To see this, in the left panel, we show the face area distribution for the 1965 map, when minor roads are excluded. From the cumulative distribution it is possible to see that the tail of the distribution is exponential.

Street length and face area distribution are robust features emerging from our city definition. In the middle panels of Fig.6, we measure the total length of the street network $L(N)$ as a function of the number of the intersections N . In [11], this measure is shown to be sub-linear with an exponent close to 0.5. Such a measure is performed over different cities, assuming the city to be an ergodic-like system. In the left panel, we show that measuring this quantity in the GLA area, we also obtain a

sub-linear trend with exponent 0.68. Nevertheless when we perform this measure on the London area as defined above, the behaviour becomes linear.

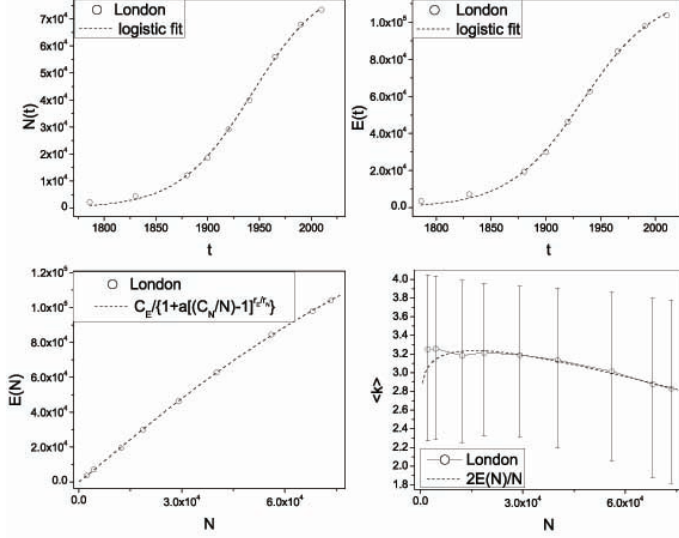


FIG. 7. Top-left panel: The number of street intersections $N(t)$ as a function of time. Top-right panel: The number of street segments $E(t)$ as a function of time. Bottom-left panel: The number of street segments $E(N)$ as a function of the number of street intersections ($R^2 = 0.9998$). Bottom-right panel: The average degree $\langle k(N) \rangle$ of the networks as a function of the number of street intersections ($R^2 = 0.7404$).

The next step in our analysis is to examine the properties of the planar graph representing London from 1786 to 2010. A planar graph is defined as a set of vertices and edges embedded in a Euclidean surface where the edges never cross each other [12]. In a city, there are bridges and tunnels so that the planarity of the graph is violated, but still the percentage of such violations is so small that the planar graph representation has to be considered acceptable.

In the top panels of Fig.7, we present measures of the number of vertices $N(t)$ and the number of edges $E(t)$ as a function of time t . Interestingly enough both the plots are well fitted by a logistic function:

$$f(t) = \frac{C}{1 + e^{-r(t-t_0)}}, \quad (1)$$

where r is the growth rate and C the carrying capacity, while t_0 is the inflection point, that is $\partial^2 f / \partial t^2|_{t=t_0} = 0$.

The logistic function was introduced in 1838 [13] to solve a problem of population growth with competing resources. It is derived from a simple differential equation with two parameters, while t_0 is just related to the shape of the curve. Hence we can frame the growth of the London street network as a very simple growth process where competition for space occurs. Such competition can be more easily understood if we consider that in UK,

particularly in London, urban sprawl is highly controlled by cordons of open space, *green belts* which in London's case goes back to 1953 [14].

The main graph measures can be easily calculated using these the logistic functions. First Eq.1 allows us to forecast the asymptotic value for the number of intersections as $N^\infty = C_N \approx 85123$, for the street segments $E^\infty = C_E \approx 115615$ and for the related average street connectivity $\langle k \rangle^\infty \approx 2.72$ in London.

In the bottom left panel of Fig.7, we show the growth of the links or segments as a function of the number of vertices, i.e. the growth of the graph. This can be expressed as:

$$E(N) = \frac{C_E}{\left[1 + a \left(\frac{C_N}{N} - 1\right)^{\frac{r_E}{r_N}}\right]}, \quad (2)$$

where $a = \exp[r_E(t_{0E} - t_{0N})]$ is constant. With the best fitting parameters $r_E/r_N \approx 1.07$, we have a good approximation to Eq.2 as the linear relation $E(N) \sim C_E N / C_N$.

In the bottom-right panel of Fig.7, we show the average degree $\langle k(N) \rangle$ of the network as a function of the number of vertices N . This is a slightly decreasing function of time, indicating that the city tends to become a more tree-like structure. Considering that $\langle k(N) \rangle = 2E/N$, this is readily computed from Eq.2.

Understanding urban growth, particularly the growth of large cities like London, is central to many perspectives on how we must design and manage urban environments in order to accommodate a sustainable environment. Here we suggest that there are some useful perspectives on the city that consider urban growth to be an analytically tractable phenomenon. Using London as a case study, we show that the complexity of a city can be described by a small number of simple parameters that generate the growth of infrastructure within a limited space. This allows us to derive analytically a few key quantities and to forecast the future evolution of the street network.

It is important to underline the fact that the growth of London is strongly influenced by the Green Belt policy, formally implemented in the 1950s, but dating back to at least the 1930s, and that this policy has constrained effectively its urbanisation process. We can also speculate that the introduction of such a policy corresponds roughly to the inflection points on the logistic functions shown in Fig.7. This observation shows that city shapes and forms are highly influenced by local policies.

We consider the observations presented here to be a first step towards a fuller understanding of pattern formation in the evolution of cities, and it is thus essential that studies of different cities are now needed to explore the existence of more universal properties of urban growth.

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